Social Learning with State-Dependent Observations

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In this note, I study a variant of the canonical binary-state binary-choice social learning model (Bikhchandani et al. [1992]). An individual would like to choose an action only in the high state. When making her own decision, she observes previous decision-makers who chose the action. Importantly, the likelihood of observing the action of previous decision-maker depends on the state. I show that when observing the action is more likely in the low state, the individual faces an inference problem: does she observe many actions because the state is high and previous decision-makers had private information about this or because the state is low and previous actions are more visible. In this situation, learning is confounded (Smith and Sørensen [2000]).

Model

The state of the world is either high $s = H$ or low $s = L$. There is a common prior belief, w.l.o.g $\Pr(H) = \mathrm{PR}(L) = \frac{1}{2}$. An infinite sequence of individuals $n = 1, 2, \ldots, \infty$ arrives in an exogeneous order. Each individual n receives a private signal and computes his private belief $p_n \in (0, 1)$ using Bayes rule. Given the state $s \in \{H, L\}$, the private belief process $< p_n >$ is i.i.d. with conditional c.d.f. F_s. We assume that F_s is differentiable for $s \in \{H, L\}$, and that the densities f_s satisfy the (strict) monotone likelihood ratio property, have full support on $\mathbb{R}$ and that

$$\lim_{p \to 1} \frac{f_H(p)}{f_L(p)} = \infty, \quad \lim_{p \to 0} \frac{f_H(p)}{f_L(p)} = 0. \tag{1}$$

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We assume that

$$F_H(p) < F_L(p) \quad \text{for any} \quad p \in (0,1). \tag{2}$$

Every individual n makes a choice $a_n \in \{0,1\}$. Each individual receives a payoff of 1 if his action matches the state, and otherwise a payoff of zero. Given that an individual n chooses $a_n = 1$, nature decides if the 1-action of n is publicly observable or not. With probability p_s all individuals $m > n$ observe that $a_n = 1$. No individual can observe n's action if $a_n = 0$. Let $\rho_H < \rho_L$.[1] Let b_n be 1 if $a_n = 1$ and the 1-action of n is observable, and otherwise let b_n be zero.

Decision Problem of an Individual

Before acting, an individual n observes his private belief p_n and the history h of observable 1-actions. Let

$$q_n(h) = \frac{\Pr(h|H)}{\Pr(h|H) + \Pr(h|L)}.$$

Applying Bayes rule implies a posterior belief r_n of n in terms of p and $q(h)$ given by

$$\frac{r_n}{1-r_n} = \frac{q_n}{1-q_n}\frac{p_n}{1-p_n}.$$

(W.l.o.g.) an individual n chooses $a_n = 1$ if $r_n \geq \frac{1}{2}$. Let $l_n = \frac{1-q_n}{q_n}$.

Steady States of the Public Belief Process

Suppose that the state is H. The likelihood ratio process l_n is a martingale conditional on state H (see Doob [1953]). Also, $< l_n >$ converges almost surely to a random variable $l_\infty = \lim_{n\to\infty} l_n$ with $supp(l_\infty) \subseteq [0,\infty)$. This follows from

[1]Suppose that the decision of agents is if to order at a restaurant or not. Then, the assumption $\rho_H < \rho_L$ loosely captures the idea that the service at a low quality restaurant might be slower than that of a high quality restaurant. Thus, the visibility of previous customers (agents who chose 1) is higher to future agents. More generally, the assumption $\rho_H \neq \rho_L$ could represent any type of state-dependent visibility of previous actions.

the Martingale Convergence Theorem for nonnegative, perhabs unbounded random variables (see Breiman, Theorem 5.14). Note that ($< l_n, b_n >$) is a Markov process on $\mathbb{R} \times \{0,1\}$ with transitions $l_{n+1} = \phi(b_n, l_n)$ given by

$$\phi(1, l_n) = l_n \frac{\Pr(r_n \geq \frac{1}{2}|H, l_n)}{\Pr(r_n \geq \frac{1}{2}|L, l_n)} \frac{\rho_H}{\rho_L} \quad \text{with probability} \quad \Pr(r_n \geq \frac{1}{2}|\omega, l_n) p_H, \tag{3}$$

$$\phi(0, l_n) = l_n \frac{1 - (\Pr(r_n \geq \frac{1}{2}|H, l_n)\rho_H)}{1 - (\Pr(r_n \geq \frac{1}{2}|L, l_n)\rho_L)} \quad \text{with probability} \quad 1 - \Pr(r_n \geq \frac{1}{2}|\omega, l_n) p_H.$$

A fixed point l of (3) satisfies for all $m \in \{0,1\}$: either $\phi(m,l) = l$ or $\Pr(m|l) = 0$. Clearly, $l = 0$ is a fixed point of (3). Any interior fixed point $l^* > 0$ must satisfy

$$\begin{aligned} \Pr(r_n \geq \frac{1}{2}|H, l^*)\rho_H &= \Pr(r_n \geq \frac{1}{2}|L, l^*)\rho_L \\ \Leftrightarrow \frac{\rho_H}{\rho_L} &= \frac{\Pr(r_n \geq \frac{1}{2}|L, l^*)}{\Pr(r_n \geq \frac{1}{2}|H, l^*)}. \end{aligned} \tag{4}$$

Thus, at an interior fixed point, the inference from the private information of the previous decision-makers offsets exactly the inference from the state-dependence of observations, see (4).

Confounded Learning

Theorem 1 *When it is more likely to observe the action of a previous decision-maker in state L, i.e. $\rho_H < \rho_L$, then the public belief process $< q_n >$ has a unique interior steady state $q^* \in (0,1)$ in state H.*

Proof. Note that $r_n \geq \frac{1}{2} \Leftrightarrow \frac{1}{l_n}\frac{p_n}{1-p_n} \geq 1 \Leftrightarrow p_n \geq \frac{l_n}{1+l_n}$. Hence $\Pr(r_n \geq \frac{1}{2}|s,l) = 1 - F_s(\frac{l_n}{1+l_n})$. So, the function $\frac{\Pr(r_n \geq \frac{1}{2}|H,l)}{\Pr(r_n \geq \frac{1}{2}|L,l)} = \frac{1-F_L(\frac{l_n}{1+l_n})}{1-F_H(\frac{l_n}{1+l_n})}$ is continuous and it follows from the monotone likelihood ratio property that the function is strictly decreasing in l. So, any interior fixed point of (3) is unique. It follows from (1) and an application of l' Hospital's rule that

$$\lim_{l_n \to \infty} \frac{1 - F_L(\frac{l_n}{1+l_n})}{1 - F_H(\frac{l_n}{1+l_n})} = \lim_{p \to 1} \frac{f_L(p)}{f_H(p)} = 0. \tag{5}$$

Clearly,

$$\frac{1 - F_L(0)}{1 - F_H(0)} = 1. \tag{6}$$

It follows from the intermediate value theorem that the function $\frac{\Pr(r_n \geq \frac{1}{2}|H,l)}{\Pr(r_n \geq \frac{1}{2}|L,l)}$ is surjective on $(0, 1]$. Hence, it follows from the assumption that $\rho_H < \rho_L$ that there exists an interior fixed point l^* of (3). This finishes the proof of the theorem since $q^* = \frac{1}{1+l^*}$ is a steady state of the public belief process in state H. ■

Confounded Learning. Note that at the limit outcome l^*, agents cannot learn anything from the observations. In this sense learning is "confounded". Confounded learning can also arise when preferences of agents are heterogeneous (see Smith and Sørensen [2000]).

Remark 1 *Theorem 4 in Smith and Sørensen [2000] shows that the fixed point l^* is locally stable if the continuation functions $\phi(b, l_n)$ are strictly increasing in l_n in a neighbourhood of l^* for $b \in \{0, 1\}$ and $\phi_l(b, l^*) \neq 1$ for some b. Locally stable means that there exists an open neighbourhood of l^* such that the process converges to l^* with positive probability once it enters this open neighbourhood.*

Benchmark: No State-Dependence of Observations

When $\rho_H = \rho_L = 1$, Smith and Sørensen [2000] show that asymptotic learning is complete. This can be easily seen from the fixed point equation (4). Note that (4) does not have a solution if $\rho_H = \rho_L$ since it follows from (2) that $\frac{\Pr(r_n \geq \frac{1}{2}|L,l)}{\Pr(r_n \geq \frac{1}{2}|H,l)} = \frac{1-F_L(\frac{l_n}{1+l})}{1-F_H(\frac{l}{1+l})} < 1$ for any $l \in (0, \infty)$. Hence, the set of fixed points of (3) is just the singleton $\{0\}$, and necessarily $\mathrm{supp}(l_\infty) = \{0\}$.

References

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