A generalized intermediate value theorem

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1 Introduction

In many economic settings, equilibria arise as fixed points of a multidimensional best response mapping. In some such situations, it is difficult to show existence of a specific equilibrium with standard fixed point theorems. For example, this happens when there are multiple equilibria and the equilibrium in question is not "attracting" under the best response in the following strong sense: The best response is not a self-map on any open environment of the equilibrium. Then, for example, Brouwer's or Kakutani's fixed point theorem cannot be invoked to prove existence of the equilibrium.

In this note, we provide a generalization of the intermediate theorem to multiple dimensions. This generalization can be used in settings with such multidimensional, non-attracting equilibria to formally establish their existence. It applies under certain conditions on the best response mapping.

In Section 2, we prove an auxiliary result—an analogue of the implicit function theorem— which may be of independent interest. Then, we prove the generalized intermediate value theorem (Theorem 2).

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2 Result

Lemma 1 *Suppose* $f : [0,1] \times [0,1] \to [-1,1]$ *is a continuous function with*

$$f(r,0) < 0 \quad \textit{for all } r, \tag{1}$$

$$f(r,1) > 0 \quad \textit{for all } r. \tag{2}$$

Then, there exist continuous functions $\hat{r}, \hat{x} : [0,1] \to [0,1]$ *such that* $\hat{r}(0) = 0$, $\hat{r}(1) = 1$, *and*

$$f(\hat{r}(t), \hat{x}(t)) = 0 \quad \textit{for all } r.$$

Proof. Let F be the upper contour set of f,

$$F = \{(r,x) : f(r,x) > 0\}.$$

F is open in $X = [0,1]^2$ since f is continuous. Let E be the connected component of E containing the upper edge, $Y_R = \{(y,1) : y \in [0,1]\}$. Let $E' = E - \partial X$. Then E' is open in $\mathbb{R}^2$. Let $\overline{E}$ be the closure of E' in X. Then, $\overline{E}$ is a connected, compact subset of $\mathbb{R}^2$. This means that we can choose a homeomorphism (that is, a continuous and invertible map) $h : [0,1]^2 \to \overline{E}$. Note that the restriction of h to $(0,1)^2$ is a homeomorphism onto its image $\mathrm{im}(h((0,1)^2)) = E'$ and the restriction of h to $\partial([0,1]^2) = [0,1]^2 - (0,1)^2$ is a homeomorphism onto its image $\mathrm{im}(h(\partial([0,1]^2)) = \overline{E} - E'$

Let $x_T = \inf\{x \in [0,1] : (1,x) \in E\}$, $x_B = \inf\{x \in [0,1] : (0,x) \in E\}$. Note that $x_B, x_T \in (0,1)$ given the assumptions of Lemma 1 and since f is continuous. The continuity of f also implies $f(1,x_T) = f(0,x_B) = 0$ and

$$f(y,x) = 0 \tag{3}$$

for all $(y,x) \in \overline{E} - E'$. Now, consider the shortest path in $\partial([0,1]^2)$ between $h^{-1}(0,x_B)$ and $h^{-1}(1,x_T)$. Then, consider its image $\mathrm{im}(h(P)) \subseteq \overline{E} - E'$. Choose a homeomorphism $\phi : [0,1] \to P$. Then, denote the components of $h \circ \phi$ by $\hat{r}$ and $\hat{x}$. By construction, $(\hat{r}, \hat{x})$ maps to $\overline{E} - E'$ so that $f(\hat{r}(t), \hat{x}(t)) = 0$, given (3). Further, by construction, $(\hat{r}(0), \hat{x}(0) = (0, x_B)$ and $(\hat{r}(1), \hat{x}(1) = (1, x_T)$. Thus, $\hat{x}, \hat{r}$ satisfy the claimed properties. This finishes the proof of Lemma 1. ■

First, Lemma 1 can be viewed as an analogue of the implicit function theorem.

To see this, we restate the implicit function theorem.

Theorem 1 *(Implicit function theorem) Suppose* $f : [0,1] \times [0,1] \to [-1,1]$ *is a continuously differentiable function. If* $f(r,x) = 0$ *and* $\partial \frac{f(r,x)}{\partial x} \neq 0$*, then there exists open set* U *with* $r \in U$ *and a continuously differentiable function* $\hat{x} : U \to [0,1]$ *so that* $f(\hat{x}(r), r) = 0$ *for all* $r \in U$ *and* $\hat{x}(r) = x$*.*

Lemma 1 yields a continuous function $(x(\cdot), r(\cdot))$ that maps to solutions of $f(r,x) = 0$. Similarly, the implicit function theorem yields a curve of solutions to $f(x,r) = 0$. However, it ensures also that for each r, there is only one $x(r)$ so that $(r, x(r))$ lies on the curve. Further, it shows that the curve is not only continuous, but continuously differentiable.

The relative advantage of Lemma 1 is that it does not rely on assumptions about partial derivatives. Thus, it is a useful alternative in situations in which verification of the qualifying condition of the implicit function is difficult and differentiability of the curve of solutions is not needed.

Second, Lemma 1 implies a generalized version of the intermediate value theorem to two dimensions as follows.

Theorem 2 *Suppose that* $f : [0,1] \times [0,1] \to [0,1]$ *is continuous and satisfies* (1)-(2)*. Suppose* $g : [0,1] \times [0,1] \to [-1,1]$ *is continuous and has the property that* $g(0,x) < 0$ *if* $f(0,x) = 0$ *and* $g(1,x) > 0$ *if* $f(1,x) = 0$*. Then, there is* $(r_0, x_0) \in (0,1)^2$ *such that* $f(r_0, x_0) = g(r_0, x_0) = 0$*.*

This is because Lemma 1 implies that $g(\hat{r}(\cdot), \hat{x}(\cdot))$ is a continuous function with $g(\hat{r}(0), \hat{x}(0)) < 0$ and $g(\hat{r}(1), \hat{x}(1)) > 0$. Thus, Theorem 2 follows by an application of the (usual) intermediate value theorem.