A generalized intermediate value theorem

Mehmet Ekmekci [*] Carl Heese [†] Stephan Lauermann [‡]

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1 Introduction

In many economic settings, equilibria arise as fixed points of a multidimensional best response mapping. In some such situations, it is difficult to show existence of a specific equilibrium with standard fixed point theorems. For example, this happens when there are multiple equilibria and the equilibrium in question is not "attracting" under the best response in the following strong sense: The best response is not a self-map on any open environment of the equilibrium. Then, for example, Brouwer's or Kakutani's fixed point theorem cannot be invoked to prove existence of the equilibrium.

In this note, we provide a generalization of the intermediate theorem to multiple dimensions. This generalization can be used in settings with such multidimensional, non-attracting equilibria to formally establish their existence. It applies under certain conditions on the best response mapping.

Our result relates to the Poincare-Miranda theorem (which is known to be equivalent to Brouwer's theorem).[1] It generalizes the theorem's statement for the case of two functions, by weakening the required conditions. The logic can be easily generalized to n functions, for any $n > 0$.

[*]Boston College, Department of Economics, mehmet.ekmekci@bc.edu
[†]University of Hong Kong, Department of Economics, heese@hku.hk
[‡]University of Bonn, Department of Economics, s.lauermann@uni-bonn.de
[1]See Miranda (1940).

2 Result

Lemma 1 *Let* $f\colon [0,1]^2 \to \mathbb{R}$ *be continuous with*

$$f(r,0) < 0 \quad \textit{for all } r, \tag{1}$$
$$f(r,1) > 0 \quad \textit{for all } r. \tag{2}$$

Then the zero set $Z = f^{-1}(0)$ *contains a connected component that intersects both* $\{0\} \times [0,1]$ *and* $\{1\} \times [0,1]$.

Proof. We first note that the zero set Z is nonempty and compact: By the intermediate value theorem, for each $x \in [0,1]$, the function $f(x,\cdot)\colon [0,1] \to \mathbb{R}$ has at least one zero, ensuring $Z \neq \emptyset$. Since f is continuous, Z is closed in the compact space $[0,1]^2$ and thus compact.

If Z is connected, the result follows immediately, as Z must intersect both the left edge $\{0\} \times [0,1]$ and the right edge $\{1\} \times [0,1]$ (since $f(0,0) < 0$, $f(0,1) > 0$, and similarly for $x = 1$).

Now suppose Z is not connected. Let A be the union of all connected components of Z that intersect the left edge $\{0\} \times [0,1]$, and let $B = Z \setminus A$. Since Z is compact and its connected components are closed, both A and B are closed in Z, and hence in $[0,1]^2$.

By the normality of $[0,1]^2$, there exist disjoint open sets $U \supset A$ and $V \supset B$ in $[0,1]^2$. Let $\pi\colon [0,1]^2 \to [0,1]$ denote the projection onto the first coordinate, $\pi(x,y) = x$. Since π is an open map, the sets $X_U = \pi(U)$ and $X_V = \pi(V)$ are open in $[0,1]$.[2]

We claim that $X_U \cup X_V = [0,1]$. Indeed, for any $x \in [0,1]$, the vertical slice $\{x\} \times [0,1]$ intersects Z (by the Intermediate Value Theorem), and this intersection must lie entirely in $A \subset U$ or $B \subset V$ (since A and B partition Z). Thus, $x \in X_U$ or $x \in X_V$.

Now, observe that $0 \in X_U$ (since A intersects the left edge) and, if A does not intersect the right edge, then $1 \in X_V$ (as B-components must account for the zeros on $\{1\} \times [0,1]$). But X_U and X_V are disjoint open sets covering $[0,1]$, which contradicts the connectedness of $[0,1]$. Therefore, A must contain a connected component that intersects both the left and right edges, completing the proof. ■

[2]This is because we can write any open set in $[0,1]^2$ as the finite union of sets $H_j \times L_j$ for $j = 1, \ldots, K$ and some $K > 0$, where H_j, L_j are open in $[0,1]$, given the compactness of $[0,1]^2$. Thus, the projection of an open set is the finite union of open sets.

Theorem 1 *Let* $f, g\colon [0,1]^2 \to [-1,1]$ *be continuous functions satisfying:*

1. $f(r,0) < 0 < f(r,1)$ *for all* $r \in [0,1]$,
2. $g(0,x) < 0$ *when* $f(0,x) = 0$,
3. $g(1,x) > 0$ *when* $f(1,x) = 0$.

Then there exists $(r_0, x_0) \in (0,1)^2$ *with* $f(r_0,x_0) = g(r_0,x_0) = 0$.

Proof. By Lemma 1, $Z = f^{-1}(0)$ contains a connected component A intersecting both vertical edges. Consider $g|_A\colon A \to [-1,1]$. The boundary behavior implies:

- At points where A meets $\{0\} \times [0,1]$, we have $g < 0$,
- At points where A meets $\{1\} \times [0,1]$, we have $g > 0$.

If $0 \notin g(A)$, then $A = g^{-1}([-1,0)) \cup g^{-1}((0,1])$ wouldbe the disjoint union of two nonempty open sets, contradicting connectedness. Thus, there exists $(r_0,x_0) \in A \subset (0,1)^2$ with $g(r_0,x_0) = 0$ (and by definition of A, $f(r_0,x_0) = 0$). ■

References

Miranda, C. (1940). *Un'osservazione su un teorema di Brouwer.* Consiglio Nazionale delle Ricerche.